Report on the paper

UNIFORM REGULARITY IN THE LOW Mach NUMBER AND INVISCID LIMITS FOR THE FULL Navier-STOKES SYSTEM IN DOMAINS WITH BOUNDARIES

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The paper under review deals with the compressible non-isentropic Navier-Stokes system, set in a bounded domain $\Omega \subset \mathbb{R}^3$. This system describes the motion of a compressible heat-conducting fluid occupying the physical domain Ω. The fluid is assumed to be polytropic and to satisfy Neumann boundary conditions for the temperature function, Navier-slip boundary conditions for the velocity field. The author studies the uniform regularity of strong solutions with respect to the low Mach number and large Reynolds and Péclet numbers. As is well-known, these are physical adimensional parameters which encode some physical characters of the fluid under consideration: the Mach number is associated to the incompressibility of the fluid, the Reynolds number with the importance of viscous friction and the Péclet number with the thermal diffusivity property. Finding uniform estimates in a suitable functional framework is important, also because it allows to address corresponding singular limit problems.

Previous studies for the system under consideration here concerned the low Mach number limit only, and dealt only with the case of well-prepared initial data. This latter assumption means that the initial data are supposed to satisfy not only suitable uniform bounds in the small parameters appearing in the equations (in the specific case of those works, there is only one small parameter, which is the Mach number), but also some structural assumptions: generally speaking, this means that the initial data already belong to the kernel of the singular perturbation operator, thus somehow killing the propagation of fast waves (acoustic waves for the low Mach number limit) and making the problem of computing the singular limit much easier. Previous works (see references [1] and [2] by Alazard) on the ill-prepared data case only concerned the inviscid system (no viscosity, no thermal diffusion) on the whole $\mathbb{R}^3$ and on exterior domains. In particular, the system being inviscid, complete-slip boundary conditions were assumed on the velocity field.

Here, the author addresses several important challenges: ill-prepared initial data and Navier-slip boundary conditions for the velocity field (which in particular implies the appearing of boundary layers in the limit process). He is able to find estimates which are uniform not only in the Mach number, but also in the Reynolds and Péclet numbers. In turn, the author uses those estimates to compute the low Mach number limit in an exterior domain and the low Mach and large Reynolds numbers limit in an exerior domain. In order to prove the results, the main new ingredient of this paper is the use of *co-normal regularity spaces*. Roughly speaking, owing to the presence of the boundary $\partial\Omega$ and of non-trivial boundary conditions, one cannot expect the solution to be smooth in the whole Ω closed, because the full gradients of the solutions are not. However, the co-normal part of the gradient, namely the part which is oriented in the tangential direction of $\partial\Omega$ enjoys additional regularity, if the initial data do. This idea (reminiscent of the notion of striated regularity *à la Chemin*) was implemented first by Masmoudi and Rousset [40]-[41] in the context of well-posedness of the system, and recently applied by the same authors in collaboration with the author of the present paper in [42] to study the low Mach number limit for the isentropic (constant temperature) system. In addition to the co-normal regularity technique, many are the difficulties which arise in the analysis, and which require clever, non-trivial ideas to be solved. In this respect, I have found the introduction of the paper very well written and clear: there, the author takes the time to explain those issues and how to solve them, making the reading of the paper (and of the details of the analysis) much easier.

Conclusions

In conclusion, I consider this paper a very good one. The author deals with a very important and difficult problem: uniform estimates with respect to several parameters of fluid systems in a bounded domain, in presence of boundary layers and for ill-prepared initial data. The paper allows for a true step forward in the state-of-the-art of the subject, solving all the non-trivial difficulties arising in the analysis in an elegant way. In addition, I have found the paper very clear and nicely written (apart from a series of minor remarks, which are listed at the end of the report), despite the great number of high technicalities which the analysis requires.

For all these reasons, I strongly recommend this work for publication in *Mémoires de la Société Mathématique de France*.